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Regularity of the Inversion Problem for the Sturm–Liouville Difference Equation

II. Two-Sided Estimates for the Diagonal Value of the Green Function

N. A. Chernyavskaya

Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva, 84105, Israel

and

L. A. Shuster

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan, 52900, Israel

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This is the second part of a study of the inversion for a Sturm–Liouville difference equation. Our main result consists in getting two-sided (sharp by order) estimates for the diagonal value of the Green difference function © 2001 Academic Press

1. INTRODUCTION

We continue the investigation of the inversion problem in $L_p(h)$ for the Sturm-Liouville difference equation (see [5] for part I),

$$-h^{-2}\Delta^{(2)}y_n + q_n(h)y_n = f_n(h), \qquad n \in Z \equiv \{0, \pm 1, \pm 2, \dots\}. \tag{1.1}$$

Here and in the sequel $h \in (0, h_0]$, h_0 is a given positive number, $f \overset{def}{=} \{f_n(h)\}_{n\in Z} \in L_p(h)$, $p \in [1, \infty)$,

$$L_p(h) \overset{def}{=} \{f : \|f\|_{L_p(h)} < \infty\}, \qquad \|f\|^p_{L_p(h)} = \sum_{n\in Z} |f_n(h)|^p h,$$

$$\Delta^{(2)}y_n \overset{def}{=} y_{n+1} - 2y_n + y_{n-1}, \qquad n \in Z$$

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and

$$0 \le q_n(h) < \infty, \qquad \sum_{k=-\infty}^{n} q_k(h) > 0,$$
$$\sum_{k=n}^{\infty} q_k(h) > 0, \quad \text{for any } n \in Z. \tag{1.2}$$

As shown in [5], if (1.2) is replaced by a stronger condition

$$\inf_{h\in(0,\,h_0]} \inf_{n\in Z} q_n(h) \stackrel{def}{=} \varepsilon > 0, \tag{1.3}$$

then for $p \in [1, \infty)$, regardless of $h \in (0, h_0]$, the following assertions hold:

(I) For any $f \in L_p(h)$, Eq. (1.1) has a unique solution $y \stackrel{def}{=} \{y_n(h)\}_{n\in Z} \in L_p(h)$ and

$$y = (Gf)(h) \stackrel{def}{=} \{(Gf)_n(h)\}_{n\in Z},$$
$$(Gf)_n(h) = \sum_{m\in Z} G_{n,m}(h) f_m(h) h, \quad n \in Z. \tag{1.4}$$

(II) Let

$$G_0 \stackrel{def}{=} \sup_{h\in(0,\,h_0]} \|G\|_{L_p(h)\to L_p(h)} < \infty. \tag{1.5}$$

Here $G_{n,m}(h)$, $n, m \in Z$ is the Green difference function corresponding to (1.1),

$$G_{n,m}(h) = \begin{cases} u_n(h)v_m(h), & n \ge m \\ u_m(h)v_n(h), & n \le m \end{cases} \tag{1.6}$$

and $\{u_n(h), v_n(h)\}_{n\in Z}$ is a special fundamental system of solutions (FSS) of the equation

$$h^{-2}\Delta^{(2)} z_n = q_n(h) z_n, \qquad n \in Z \tag{1.7}$$

(see Sect. 2). If (I)–(II) hold, we say that the inversion problem for (1.1) is regular in $L_p(h)$ (see [5]). Recall (see [5]) that our goal is to obtain a criterion for (I)–(II) to hold for those equations (1.1) whose potential $q \stackrel{def}{=} \{q_k(h)\}_{n\in Z}$ is not separated from zero (in the sense of (1.2)). We treat the above problem in our forthcoming paper [6]. Our approach builds upon the following assertions:

THEOREM 1.1 [5]. *For any* $p \in [1, \infty)$ *the inversion problem for* (1.1) *is regular in* $L_p(h)$ *provided*

$$H < \infty, \qquad H \stackrel{def}{=} \sup_{h\in(0,\,h_0]} \sup_{n\in Z} \sum_{m=-\infty}^{\infty} G_{n,m}(h) h. \tag{1.8}$$

THEOREM 1.2 [5]. *Let* $\rho_n(h) \overset{def}{=} u_n(h)v_n(h)$, $n \in Z$. *Then for* $n \neq m$ *the Green function* $G_{n,m}(h)$ *admits a representation of the Davies–Harrell type* [7]:

$$G_{n,m}(h) = \begin{cases} \sqrt{\rho_n(h)\rho_m(h)}\prod_{k=n}^{m-1}\Big[1 + \frac{u_k(h)}{u_{k+1}(h)}\frac{h}{\rho_k(h)}\Big]^{-1/2}, & n < m \\ \sqrt{\rho_n(h)\rho_m(h)}\prod_{k=m}^{n-1}\Big[1 + \frac{u_k(h)}{u_{k+1}(h)}\frac{h}{\rho_k(h)}\Big]^{-1/2}, & n > m. \end{cases} \tag{1.9}$$

In (1.9), $u = \{u_n(h)\}_{n\in Z}$ stands for a non-increasing solution of (1.7). In view of Theorem 1.1, it is clear that to check condition (1.8), one needs to estimate the Green function. We do not manage to obtain such inequalities using (1.6) because one does not know any a priori (sharp by order) estimate for each solution $\{u, v\}$. Therefore, as in [2, 4], we use (1.9) instead of (1.6). The main advantage of (1.9) in contrast to (1.6) is as follows: (1.9) reduces the problem of estimating $G_{n,m}(h)$ for $n \neq m$ to the problem of estimating the diagonal value $G_{n,m}(h) = \rho_n(h)$, $n \in Z$; i.e., one needs to estimate the product of solutions and not each of them separately. We manage to obtain the required inequalities for the product of solutions. This allows us to estimate $G_{n,m}(h)$ for $n \neq m$, and, finally, to verify (1.8) [3, 5, 6]. Our main result consists in getting two-sided (sharp by order) estimates for the diagonal value of the Green function (1.6) (see Theorem 1.3 below). To state the result, let us introduce auxiliary functions

$$\ell_n(h) = \begin{cases} 0, & \text{if } q_n(h)h^2 \geq 1 \\ \min_{j\geq 0}\Big\{j : j\sum_{k=n-j}^{n+j} q_k(h)h^2 \geq 1\Big\}, & \text{if } q_n(h)h^2 < 1 \end{cases} \tag{1.10}$$

$$d_n(h) = \begin{cases} \frac{h}{1+q_n(h)h^2}, & \text{if } \ell_n(h) = 0 \\ \ell_n(h)h, & \text{if } \ell_n(h) \neq 0. \end{cases} \tag{1.11}$$

The functions $\ell_n(h)$, $d_n(h)$ were first introduced in [1].

THEOREM 1.3. *For* $n \in Z$ *one has*

$$8^{-1}d_n(h) \leq \rho_n(h) \leq 16 d_n(h), \qquad h \in (0, h_0]. \tag{1.12}$$

In view of Theorem 1.3, let us emphasize that inequalities (1.12) do not give any information concerning the inversion of (1.1) in $L_p(h)$. The same can be said about formula (1.9). It is peculiar that two results with such a negative property generate a useful tool for investigation of (1.1) if brought together. This is confirmed by the main result of our study, Theorem 1.4. Its proof is given in [6] and only builds upon Theorems 1.1–1.3 and the properties of the functional $d_n(h)$.

THEOREM 1.4 [6]. *Let* $p \in [1, \infty)$. *The inversion problem in* $L_p(h)$ *for Eq.* (1.1) *is regular if and only if*

$$A(q) < \infty, \qquad A(q) \overset{def}{=} \sup_{h\in(0,h_0]}\sup_{n\in Z} d_n(h). \tag{1.13}$$

Note that Theorem 1.4 can serve as a main example of applying Theorem 1.3. We now give some comments on condition (1.13). It is not an easy problem to compute $d_n(h)$, $n \in Z$, but usually one does not need to know all values of the functional $d_n(h)$ in order to verify whether (I)–(II) hold. As a rule, in order to estimate the functional $A(q)$ one instead uses some estimates for the values of the functional $\ell_n(h)$ (and hence of $d_n(h)$—see (1.10)–(1.11)) at the local maxima. The last remark often simplifies the application of (1.13). We systematically stress that in the examples below.

EXAMPLE 1. Consider (1.1) with potential $q = \{q_n(h)\}_{n\in Z}$ satisfying condition (1.3). Denote

$$\begin{aligned} Z_1(h) &= \{n \in Z : q_n(h)h^2 \geq 1\}, \\ Z_2(h) &= \{n \in Z : q_n(h)h^2 < 1\}. \end{aligned} \tag{1.14}$$

Clearly, $Z = Z_1(h) \cup Z_2(h)$, $Z_1(h) \cap Z_2(h) = \emptyset$ for $h \in (0, h_0]$. If $n \in Z_1(h)$, by (1.10)–(1.11) we get

$$d_n(h) = \frac{h}{1 + q_n(h)h^2} \leq \frac{h}{2} \leq h_0 \Rightarrow \sup_{h\in(0,\, h_0]} \sup_{n\in Z_1(h)} d_n(h) \leq h_0. \tag{1.15}$$

Let $n \in Z_2(h)$. To estimate $\ell_n(h)$, set $j_0 = 1 + [1/\sqrt{2\varepsilon} \cdot h]$. Then for all $n \in Z_2(h)$ we get

$$j_0 \sum_{k=n-j_0}^{n+j_0} q_k(h)h^2 \geq j_0 \sum_{k=n-j_0}^{n+j_0} \varepsilon h^2 = j_0(2j_0 + 1)\varepsilon h^2 > 2\varepsilon h^2 j_0^2 > 1.$$

Hence $\ell_n(h) \leq j_0$ for all $n \in Z_2(h)$. Therefore

$$\begin{aligned} \sup_{h\in(0,\, h_0]} \sup_{n\in Z_2(h)} d_n(h) &= \sup_{h\in(0,\, h_0]} \sup_{n\in Z_2(h)} \ell_n(h)h \\ &\leq \sup_{h\in(0,\, h_0]} \left\{1 + \left[\frac{1}{\sqrt{2\varepsilon h}}\right]\right\} h \leq \sup_{h\in(0,\, h_0]} \left(1 + \frac{1}{\sqrt{2\varepsilon}}\, \frac{1}{h}\right) h \\ &\leq h_0 + \frac{1}{\sqrt{2\varepsilon}} \Rightarrow A(q) < \infty. \end{aligned}$$

Thus under condition (1.3), the inversion problem for (1.1) is regular in $L_p(h)$ for all $p \in [1, \infty)$ (see [5]). Here we used a uniform estimate of $d_n(h)$ for all $n \in Z$ instead of exact values of $d_n(h)$.

EXAMPLE 2. Consider Eq. (1.1) with potential $q = \{q_n(h)\}_{n\in Z}$ where

$$q_n(h) = \begin{cases} |n|h, & \text{if } n = 2k,\ k \in Z, \\ 0, & \text{if } n = 2k+1,\ k \in Z, \end{cases} \quad h \in (0, h_0],\ h_0 = \tfrac{1}{2}. \tag{1.16}$$

Clearly, $q_n(h) = q_{-n}(h)$ and therefore one only needs to estimate $d_n(h)$ for $n \geq 0$. In this example it is not hard to estimate $\rho_n(h)$ for all $n \gg 1$. Indeed, one has $q_{2n}(h)h^2 \geq 1$ for $n \geq (2h^3)^{-1}$ and therefore by (1.11) and (1.16) we get

$$d_{2n}(h) = \frac{h}{1+2nh^3} \Rightarrow \frac{1}{8}\,\frac{h}{1+2nh^3} \leq \rho_n(h) \leq \frac{16h}{1+2nh^3} \text{ for } n \geq (2h^3)^{-1}.$$

In addition, $\ell_{2n+1}(h) = 1$ for $n \geq (2h^3)^{-1}$ since $q_{2n+1}(h)h^2 = 0 < 1$, and

$$1 \cdot \sum_{k=(2n+1)-1}^{(2n+1)+1} q_k(h)h^2 = q_{2n}(h)h^2 + q_{2n+2}(h)h^2 \geq 4nh^3 \geq 2 \geq 1$$

$$\Rightarrow d_n(h) = \ell_{2n+1}(h)h = h \Rightarrow \frac{h}{8} \leq \rho_{2n+1}(h) \leq 16h$$

for $n \geq (2h^3)^{-1}$. Thus $\sup_{n \geq (2h^3)^{-1}} d_n(h) \leq 16h_0 = 8$ for any fixed $h \in (0, h_0]$. We now consider the values $\ell_k(h)$ for $0 \leq k < (2h^3)^{-1}$. For all such k's one has $\ell_k(h) \neq 0$ since $q_k(h)h^2 < 1$. Let us show that the function $\ell_k(h)$ attains its maximum at $k = 0$. Indeed, from (1.16) it easily follows that

$$\ell_0(h) \sum_{s=n-\ell_0(h)}^{n+\ell_0(h)} q_s(h)h^2 \geq \ell_0(h) \sum_{k=-\ell_0(h)}^{\ell_0(h)} q_s(h)h^2 \geq 1, \quad n \in Z. \tag{1.17}$$

Therefore $\ell_0(h) \geq \ell_n(h),\ n \in Z$ in view of (1.10). To estimate $\ell_0(h)$, set $j_0 = 1 + [h^{-1}]$. Then by (1.16) we get

$$2j_0 \sum_{k=-2j_0}^{2j_0} q_k(h)h^2 = 4j_0 \sum_{k=1}^{2j_0} q_k(h)h^2$$

$$= 4j_0 h^3[2 + 4 + \cdots + 2j_0] \geq (j_0 h)^3 \geq 1.$$

Hence $\ell_0(h) \leq j_0$ and therefore $d_0(h) = \ell_0(h)h \leq j_0 h \leq 2$. We conclude that $A(q) \leq 8 < \infty$ and the inversion problem for (1.1) with potential (1.16) is regular for all $p \in [1, \infty)$.

EXAMPLE 3. Consider Eq. (1.1) with potential $q = \{q_n(h)\}_{n \in Z}$ where

$$q_n(h) = \begin{cases} 0, & \text{if } |n| \neq k^2,\ k \in Z, \\ |n|h, & \text{if } |n| = k^2,\ k \in Z, \end{cases} \quad h \in (0, h_0],\ h_0 = 1. \tag{1.18}$$

As in Example 2, in order to illustrate Theorem 1.3, we compute $d_n(h)$ for $|n| \gg 1$. Clearly, $q_n(h) = q_{-n}(h)$, $n \in Z$, and therefore we restrict ourselves by $n \gg 1$. Fix $h \in (0, h_0]$. For $k \geq h^{-3/2}$ we get $q_{k^2}(h)h^2 = k^2h^3 \geq 1$. By (1.11), we conclude

$$d_{k^2}(h) = \frac{h}{1+k^2h} \Rightarrow \frac{8^{-1}h}{1+k^2h^3} \leq \rho_{k^2}(h) \leq \frac{16h}{1+k^2h^3} \quad \text{for } k \geq h^{-3/2}.$$

Let us compute $d_{k^2+s}(h)$, $s = \overline{1, 2k}$ for $k \geq h^{-3/2}$. Since $q_{k^2+s}(h) = 0$ for all such k's, one has $\ell_{k^2+s}(h) \neq 0$. Let $s = \overline{1, k}$. Then by (1.17), one has

$$(s-1) \sum_{\ell=k^2+s-(s-1)}^{k^2+s+(s-1)} q_\ell(h)h^2 = (s-1) \cdot 0 = 0 < 1$$

$$s \sum_{\ell=(k^2+s)-s}^{(k^2+s)+s} q_\ell(h)h^2 \geq sk^2h^3 \geq 1.$$

Hence $d_{k^2+s}(h) = sh$ for $s = \overline{1, k}$, $k \geq h^{-3/2}$ since $\ell_{k^2+s}(h) = s$ (see (1.10)–(1.11)). Similarly, one can verify that $d_{k^2+s} = (2k+1-s)h$ for $s = \overline{k+1, 2k}$. This implies that for $k \geq h^{-3/2}$, $s = \overline{1, 2k}$, one has

$$8^{-1} d_{k^2+s}(h) \leq \rho_{k^2+s}(h) \leq 16 d_{k^2+s}(h),$$
$$d_{k^2+s}(h) = \begin{cases} sh, & \text{if } s = \overline{1, k} \\ (2k+1-s)h, & \text{if } s = \overline{k+1, 2k} \end{cases}. \tag{1.19}$$

In addition, by (1.19) at the points $k^2 + k$, $k^2 + k + 1$ the function $d_{k^2+s}(h)$ has local maximum equal to kh. Hence $d_{k^2+k}(h) \to \infty$ as $k \to \infty$ for any $h \in (0, 1]$. This means that the inversion problem for (1.1) with potential (1.18) is not regular for all $p \in [1, \infty)$ since $A(q) = \infty$.

Finally, we note that the present paper is a natural continuation of [1–4, 8]. We systematically use and strengthen methods and devices of the papers cited above.

2. PRELIMINARIES

In the sequel we assume $h \in (0, h_0]$, h_0 is a fixed positive number. We always assume that (1.2) holds.

THEOREM 2.1 [5]. *There exists a FSS* $\{u, v\} = \{u_n(h), v_n(h)\}_{n \in Z}$ *of* (1.7) *such that*

$$0 < u_{n+1}(h) \leq u_n(h), \qquad v_{n+1}(h) \geq v_n(h) > 0, \quad n \in Z$$
$$v_{n+1}(h)u_n(h) - u_{n+1}(h)v_n(h) = h,$$
$$u_n(h) = v_n(h) \sum_{k=n}^{\infty} \frac{h}{v_k(h)v_{k+1}(h)} \tag{2.1}$$
$$\lim_{n \to -\infty} \frac{v_n(h)}{u_n(h)} = \lim_{n \to \infty} \frac{u_n(h)}{v_n(h)} = 0.$$

DEFINITION 2.1 [5]. An FSS of (1.7) with properties (2.1) is called a principal FSS (PFSS).

3. ESTIMATES FOR "LOGARITHMIC DERIVATIVES" OF PFSS AND THEIR APPLICATIONS

In this section we obtain two-sided (sharp by order) a priori estimates for "logarithmic derivatives" of PFSS of (1.7) [2, 4]:

$$\frac{v_{n+1}(h) - v_n(h)}{hv_n(h)}, \qquad \frac{u_{n-1}(h) - u_n(h)}{hu_n(h)}, \quad n \in Z. \tag{3.1}$$

As in [2, 4], these inequalities imply estimates for $\rho_n(h)$ expressed in terms of one-sided auxiliary functions $d_n^{(1)}(h)$ and $d_n^{(2)}(h)$. Let us introduce these functions. For $n \in Z$ set

$$\ell_n^{(1)}(h) = \begin{cases} 0, & \text{if } q_n(h)h^2 \geq 1 \\ \min_{j\geq 1}\{j : j\sum_{k=0}^{j-1} q_{n-k}(h)h^2\} \geq 1, & \text{if } q_n(h)h^2 < 1, \end{cases} \tag{3.2}$$

$$\ell_n^{(2)}(h) = \begin{cases} 0, & \text{if } q_n(h)h^2 \geq 1 \\ \min_{j\geq 1}\{j : j\sum_{k=0}^{j-1} q_{n+k}(h)h^2\} \geq 1, & \text{if } q_n(h)h^2 < 1 \end{cases} \tag{3.3}$$

$$d_n^{(1)}(h) = \begin{cases} \frac{h}{1+q_n(h)h^2}, & \text{if } \ell_n^{(1)}(h) = 0 \\ \ell_n^{(1)}(h) \cdot h, & \text{if } \ell_n^{(1)}(h) \neq 0 \end{cases} \tag{3.4}$$

$$d_n^{(2)}(h) = \begin{cases} \frac{h}{1+q_n(h)h^2}, & \text{if } \ell_n^{(2)}(h) = 0 \\ \ell_n^{(2)}(h)h, & \text{if } \ell_n^{(2)}(h) \neq 0. \end{cases} \tag{3.5}$$

The next assertion is the main result of this section.

THEOREM 3.1. *For $n \in Z$ one has*

$$\frac{1}{2d_n^{(1)}(h)} \leq \frac{v_{n+1}(h) - v_n(h)}{hv_n(h)} \leq \frac{4}{d_n^{(1)}(h)} \tag{3.6}$$

$$\frac{1}{2d_n^{(2)}(h)} \leq \frac{u_{n-1}(h) - u_n(h)}{hu_n(h)} \leq \frac{4}{d_n^{(2)}(h)} \tag{3.7}$$

$$\frac{d_n^{(1)}(h) \cdot d_n^{(2)}(h)}{4(d_n^{(1)}(h) + d_n^{(2)}(h)} \leq \rho_n(h) \leq \frac{4d_n^{(1)}(h) \cdot d_n^{(2)}(h)}{d_n^{(1)}(h) + d_n^{(2)}(h)}. \tag{3.8}$$

Proof. Inequalities (3.6)–(3.7) are proved in the same way; therefore we check only (3.6). From (2.1) for $q_n(h)h^2 \geq 1$ we deduce (3.6)–(3.7),

$$\begin{aligned} v_{n+1}(h) - v_n(h) &= v_n(h) - v_{n-1}(h) + q_n(h)h^2 v_n(h) \\ &\leq v_n(h)(1 + q_n(h)h^2) = v_n(h)\frac{h}{d_n^{(1)}(h)} \end{aligned} \tag{3.9}$$

$$\begin{aligned} v_{n+1}(h) - v_n(h) &= v_n(h) - v_{n-1}(h) + q_n(h)h^2 v_n(h) \\ &\geq v_n(h)h^2 q_n(h) \geq \frac{v_n(h)h}{2d_n^{(1)}(h)}. \end{aligned}$$

As above, let $q_n(h)h^2 \geq 1$. We check (3.8). From (2.1) we get

$$\frac{v_{n+1}(h) - v_n(h)}{hv_n(h)} + \frac{u_n(h) - u_{n+1}(h)}{hu_n(h)} = \frac{1}{u_n(h)v_n(h)} = \frac{1}{\rho_n(h)}. \tag{3.10}$$

From (1.7), (2.1), and the inequality $\Delta^{(2)}u_n \geq 0, \;\; n \in Z$, it follows that

$$\begin{aligned} (1 + q_n(h)h^2)u_n &= u_{n-1} - u_n + u_{n+1} \geq u_{n-1} - u_n \\ &\geq u_n - u_{n+1}, \qquad n \in Z. \end{aligned} \tag{3.11}$$

Then (3.9), (3.10), and (3.11) imply the lower estimate of (3.8):

$$2\frac{1 + q_n(h)h^2}{h} \geq \frac{v_{n+1}(h) - v_n(h)}{hv_n(h)} + \frac{u_n(h) - u_{n+1}(h)}{hu_n(h)} = \frac{1}{\rho_n(h)}.$$

Similarly, using the inequalities proved above we derive the upper estimate of (3.8):

$$\frac{1}{\rho_n(h)} > \frac{v_{n+1}(h) - v_n(h)}{hv_n(h)} \geq q_n(h)h \geq \frac{1 + q_n(h)h^2}{2h}.$$

Thus we obtain estimates (3.8) in the case where $q_n(h)h^2 \geq 1$.

To treat the case $q_n(h)h^2 < 1$, we need some auxiliary assertions.

LEMMA 3.1. *Let $n \in Z, \;\; j \geq 1$. Then one has*

$$\begin{aligned} j(v_{n+1}(h) - v_n(h)) &= v_n(h) - v_{n-j}(h) \\ &\quad + \sum_{s=0}^{j-1}\sum_{k=0}^{s} q_{n-k}(h)v_{n-k}(h)h^2, \end{aligned} \tag{3.12}$$

$$\begin{aligned} j(u_{n-1}(h) - u_n(h)) &= u_n(h) - u_{n+j}(h) \\ &\quad + \sum_{s=0}^{j-1}\sum_{k=0}^{s} q_{n+k}(h)u_{n+k}(h)h^2. \end{aligned} \tag{3.13}$$

Proof. Both equalities are proved in the same way. Let us check, say, (3.12). We sum equalities (3.14) over $k = 0, \ldots, s, \quad s \geq 0$ and obtain (3.15),

$$v_{n-k+1}(h) - v_{n-k}(h) = v_{n-k}(h) - v_{n-k-1}(h) + q_{n-k}(h) v_{n-k}(h) h^2. \tag{3.14}$$

$$v_{n+1} - v_n(h) = v_{n-s}(h) - v_{n-s-1}(h) + \sum_{k=0}^{s} q_{n-k}(h) v_{n-k}(h) h^2. \tag{3.15}$$

By summing equalities (3.15) over $s = 0, 1, \ldots, j-1$ we get (3.12). ∎

Estimates (3.6) and (3.7) are proved in the same way. Let us check, say, (3.6). Throughout the sequel we assume $q_n(h)h^2 < 1$. Clearly, one then has $\ell_n^{(1)}(h) \geq 2$. Indeed, otherwise (3.2) would imply for $j = j_0 = 1$,

$$j_0 \sum_{k=0}^{j_0-1} q_{n-k}(h) h^2 = q_n(h) h^2 < 1 \Rightarrow \ell_n^{(1)}(h) > j_0 \Rightarrow \ell_n(h) \geq 2.$$

Set $j_0 = \ell_n^{(1)}(h) - 1$. From (3.12), (2.1), and (3.2), we get

$$\begin{aligned} j_0(v_{n+1}(h) - v_n(h)) &= v_n(h) - v_{n-j_0}(h) + \sum_{s=0}^{j_0-1} \sum_{k=0}^{s} q_{n-k}(h) v_{n-k}(h) h^2 \\ &< v_n(h) + v_n(h) \sum_{s=0}^{j_0-1} \sum_{k=0}^{s} q_{n-k}(h) h^2 \\ &\leq v_n(h) + v_n(h) \left[\sum_{k=0}^{j_0-1} q_{n-k}(h) h^2 \right] \sum_{s=0}^{j_0-1} 1 \\ &= v_n(h) \left[1 + j_0 \sum_{s=0}^{j_0-1} q_{n-k}(h) h^2 \right] \leq 2 v_n(h). \end{aligned}$$

The latter inequality implies the upper estimate of (3.6),

$$\frac{v_{n+1}(h) - v_n(h)}{h} \leq \frac{2}{j_0 h} = \frac{2}{(\ell_n(h) - 1)h} \leq \frac{4}{\ell_n(h) h} = \frac{4}{d_n^{(1)}(h)}.$$

Let us prove the lower estimate of (3.6). Let $j_0 = 2\ell_n^{(1)}(h) - 1$. From (3.12) and (2.1) we get

$$
\begin{aligned}
j_0(v_{n+1}(h)-v_n(h)) &= v_n(h)-v_{n-j_0}(h)+\sum_{s=0}^{j_0-1}\sum_{k=0}^{s} q_{n-k}(h)v_{n-k}(h)h^2 \\
&\geq v_n(h)-v_{n-j_0}(h)+v_{n-j_0+1}(h)\sum_{s=0}^{j_0-1}\sum_{k=0}^{s} q_{n-k}(h)h^2 \\
&\geq v_n(h)-v_{n-j_0}(h)+v_{n-j_0}(h)\sum_{s=0}^{j_0-1}\sum_{k=0}^{s} q_{n-k}(h)h^2. \qquad (3.16)
\end{aligned}
$$

To estimate the sum in (3.16), we make use of the definition of $\ell_n^{(1)}(h)$,

$$
\begin{aligned}
\sum_{s=0}^{j_0-1}\sum_{k=0}^{s} q_{n-k}(h)h^2 &= \sum_{s=0}^{2\ell_n(h)-2}\sum_{k=0}^{s} q_{n-k}(h)h^2 \geq \sum_{s=\ell_n^{(1)}(h)-1}^{2\ell_n^{(1)}(h)-2}\sum_{k=0}^{s} q_{n-k}(h)h^2 \\
&\geq \left(\sum_{k=0}^{\ell_n^{(1)}(h)-1} q_{n-k}(h)h^2\right)\sum_{s=\ell_n^{(1)}(h)-1}^{2\ell_n^{(1)}(h)-2} 1 \\
&= \ell_n^{(1)}(h)\sum_{k=0}^{\ell_n^{(1)}(h)-1} q_{n-k}(h)h^2 \geq 1. \qquad (3.17)
\end{aligned}
$$

Now (3.16) and (3.17) imply

$$
\begin{aligned}
\big(2\ell_n^{(1)}(h)-1\big)(v_{n+1}(h)-v_n(h)) &> v_n(h)-v_{n-j_0}(h)+v_{n-j_0}(h) \\
&= v_n(h).
\end{aligned}
$$

We thus obtain the lower estimate of (3.6) since

$$
\frac{v_{n+1}(h)-v_n(h)}{hv_n(h)} > \frac{1}{(2\ell_n^{(1)}(h)-1)h} \geq \frac{1}{2\ell_n^{(1)}(h)h} = \frac{1}{2d_n^{(1)}(h)}.
$$

Thus it remains to check (3.8) in the case $q_n(h)h^2 < 1$ (the case $q_n(h)h^2 \geq 1$ has already been treated above). From (3.6), (3.7), and (2.1) we deduce (3.18), (3.19), and (3.20):

$$
1+\frac{1}{2\ell_n^{(1)}(h)} \leq \frac{v_{n+1}(h)}{v_n(h)} \leq 1+\frac{4}{\ell_n^{(1)}(h)} \qquad (3.18)
$$

$$
1+\frac{1}{2\ell_n^{(2)}(h)} \leq \frac{u_{n-1}(h)}{u_n(h)} \leq 1+\frac{4}{\ell_n^{(2)}(h)} \qquad (3.19)
$$

$$
\frac{v_{n+1}(h)}{v_n(h)} - \frac{u_{n+1}(h)}{u_n(h)} = \frac{h}{\rho_n(h)}. \qquad (3.20)
$$

In addition, since $h^{-2}\Delta^{(2)}u_n = q_n(h)u_n$, one has

$$\frac{u_{n-1}(h)}{u_n(h)} = 2 + q_n(h)h^2 - \frac{u_{n+1}(h)}{u_n(h)}. \tag{3.21}$$

Therefore, (3.19) and (3.21) imply

$$1 + q_n(h)h^2 - \frac{4}{\ell_n^{(2)}(h)} \le \frac{u_{n+1}(h)}{u_n(h)} \le 1 + q_n(h)h^2 - \frac{1}{2\ell_n^{(2)}(h)}. \tag{3.22}$$

We now plug estimates (3.18) and (3.22) into (3.20):

$$\begin{aligned}\frac{h}{\rho_n(h)} &\ge 1 + \frac{1}{2\ell_n^{(1)}(h)} - \left[1 + q_n(h)h^2 - \frac{1}{2\ell_n^{(2)}(h)}\right]\\ &= \frac{1}{2}\left[\frac{1}{\ell_n^{(1)}(h)} + \frac{1}{\ell_n^{(2)}(h)}\right] - q_n(h)h^2.\end{aligned}$$

The latter inequality immediately implies

$$\frac{\rho_n(h)}{1 + q_n(h)h\rho_n(h)} \le \frac{2d_n^{(1)}(h)d_n^{(2)}(h)}{d_n^{(1)}(h) + d_n^{(2)}(h)}. \tag{3.23}$$

Let us now verify that $q_n(h)h\rho_n(h) \le 1,\ \ n \in Z$. From Theorem 2.1 it follows that

$$\begin{aligned}&q_n(h)h\rho_n(h)\\ &\quad= q_n(h)h^2 v_n(h)^2 \sum_{k=n}^{\infty} \frac{1}{v_k(h)v_{k+1}(h)}\\ &\quad= v_n(h)[(v_{n+1}(h) - v_n(h)) - (v_n(h) - v_{n-1}(h))] \sum_{k=n}^{\infty} \frac{1}{v_k(h)v_{k+1}(h)}\\ &\quad\le v_n(h)(v_{n+1}(h) - v_n(h)) \sum_{k=n}^{\infty} \frac{1}{v_k(h)v_{k+1}(h)}\\ &\quad\le v_n(h) \sum_{k=n}^{\infty} \frac{v_{k+1}(h) - v_k(h)}{v_k(h)v_{k+1}(h)} = v_n(h) \sum_{k=n}^{\infty}\left[\frac{1}{v_k(h)} - \frac{1}{v_{k+1}(h)}\right]\\ &\quad\le \frac{v_n(h)}{v_n(h)} = 1.\end{aligned}$$

Together with (3.23), this implies the upper estimate of (3.8):

$$\frac{\rho_n(h)}{2} \le \frac{\rho_n(h)}{1 + q_n(h)h\rho_n(h)} \le 2\frac{d_n^{(1)}d_n^{(2)}}{d_n^{(1)}(h) + d_n^{(2)}(h)}.$$

The lower estimate of (3.8) follows from (3.18), (3.20), and (3.22):

$$\begin{aligned}\frac{h}{\rho_n(h)} &\le 1+\frac{4}{\ell_n^{(1)}(h)}-\left[1+q_n(h)h^2-\frac{4}{\ell_n^{(2)}(h)}\right]\\ &=4\left(\frac{1}{\ell_n^{(1)}(h)}+\frac{1}{\ell_n^{(2)}(n)}\right)-q_n(h)h^2\\ &\le 4h\left(\frac{1}{d_n^{(1)}(h)}+\frac{1}{d_n^{(2)}(h)}\right).\end{aligned}$$

∎

4. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 1.3. Recall that for $q_n(h)h^2 \ge 1$ estimates (1.11) coincide with (3.18) in view of the definitions of $d_n(h)$, $d_n^{(1)}$, and $d_n^{(2)}(h)$. Therefore, below we only consider the case $q_n(h)h^2 < 1$, $n \in Z$. In this case, one has $\ell_n^{(1)}(n) \ge 2$, $\ell_n^{(2)}(n) \ge 2$ (see Sect. 3), and the following inequalities hold:

$$\ell_n^{(1)}(h)\sum_{k=n-\ell_n^{(1)}(h)}^{n+\ell_n^{(1)}(h)} q_k(h)h^2 \ge \ell_n^{(1)}(h)\sum_{k=n-\ell_n^{(1)}(h)+1}^{n} q_k(h)h^2 \ge 1, \tag{4.1}$$

$$\ell_n^{(2)}(h)\sum_{k=n-\ell_n^{(2)}(h)}^{n+\ell_n^{(2)}(h)} q_k(h)h^2 \ge \ell_n^{(2)}(h)\sum_{k=n}^{n+\ell_n^{(2)}(h)-1} q_k(h)h^2 \ge 1. \tag{4.2}$$

Hence $\ell_n^{(1)}(h) \ge \ell_n(h)$, $\ell_n^{(2)}(h) \ge \ell_n(h)$ in view of (1.1). This implies

$$\begin{aligned}\frac{2}{\ell_n(h)} &\ge \frac{1}{\ell_n^{(1)}(h)}+\frac{1}{\ell_n^{(2)}(h)} \Rightarrow \frac{\ell_n^{(1)}(h)\ell_n^{(2)}(h)}{\ell_n^{(1)}(h)+\ell_n^{(2)}(h)} \ge \frac{\ell_n(h)}{2}\\ &\Rightarrow \frac{d_n^{(1)}(h)d_n^{(2)}(h)}{d_n^{(1)}(h)+d_n^{(2)}(h)} \ge \frac{d_n(h)}{2} \Rightarrow \rho_n(h)\\ &\ge \frac{1}{4}\frac{d_n^{(1)}(h)d_n^{(2)}(h)}{d_n^{(1)}(h)+d_n^{(2)}(h)} \ge \frac{d_n(h)}{8}.\end{aligned}$$

We thus obtain the lower estimate of (1.12). Let us verify the upper estimate of (1.12). Denote $m = \min\{\ell_n^{(1)}(h), \ell_n^{(2)}(h)\}$, $M = \max\{\ell_n^{(1)}(h), \ell_n^{(2)}(h)\}$, and consider two separate cases: (1) $m = 2$ and (2) $m \ge 3$.

(1) Since $q_n(h)h^2 < 1$, one has $\ell_n(h) \geq 1$. Then

$$\frac{\ell_n^{(1)}(h)\ell_n^{(2)}(h)}{\ell_n^{(1)}(h)+\ell_n^{(2)}(h)} = \frac{mM}{m+M} \leq m = 2 \leq \ell_n(h)+1 \leq 2\ell_n(h). \tag{4.3}$$

In this case (1.12) follows from (3.8) and (4.3):

$$\rho_n(h) \leq 4\frac{d_n^{(1)}(h)d_n^{(2)}(h)}{d_n^{(1)}(h)+d_n^{(2)}(h)} \leq 8d_n(h).$$

(2) If $m \geq 3$, then $\ell_n^{(1)}(h) - 2 \geq 1$, $\ell_n^{(2)}(h) - 2 \geq 1$, and by (3.2) and (3.3), one has

$$(\ell_n^{(1)}(h)-1) \sum_{k=n-\ell_n^{(1)}(h)+2}^{n} q_k(h)h^2 < 1,$$

$$(\ell_n^{(2)}(h)-1) \sum_{k=n}^{n+\ell_n^{(2)}(h)-2} q_k(h)h^2 < 1.$$

These inequalities imply

$$(m-1) \sum_{k=n-m+2}^{n} q_k(h)h^2 < 1, \qquad (m-1) \sum_{k=n}^{n+m-2} q_k(h)h^2 < 1. \tag{4.4}$$

By summing inequalities (4.4) and strengthening the resulting inequality, we obtain

$$\frac{m-1}{2} \sum_{k=n-m+2}^{n+m-2} q_k(h)h^2 < 1. \tag{4.5}$$

Let $j_0 = [\frac{m-1}{2}]$. Then $1 \leq j_0 \leq m-2$. Therefore, by strengthening (4.5), we get

$$j_0 \sum_{k=n-j_0}^{n+j_0} q_k(h)h^2 < 1 \Rightarrow \ell_n(h) \geq j_0. \tag{4.6}$$

Since $4j_0 \geq m$ for $m \geq 3$, one has $4\ell_n(h) \geq m$ and therefore

$$\frac{\ell_n^{(1)}(h)\ell_n^{(2)}(h)}{\ell_n^{(1)}(h)+\ell_n^{(2)}(h)} = \frac{mM}{m+M} \leq m \leq 4\ell_n(h) \Rightarrow \rho_n(h)$$

$$\leq 4\frac{d_n^{(1)}(h)d_n^{(2)}(h)}{d_n^{(1)}(h)+d_n^{(2)}(h)} \leq 16d_n(h). \qquad \blacksquare$$

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